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ON THE METHOD OF DISCRETE VORTICES

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Application of the method of discrete slanting horseshoe vortices to the linear steady state problem of stable flow past a wing of finite span and complex plan shape, and of a schematized aircraft with straight edges is considered. An integral equation is obtained for the vortex layer intensity on a wing. It is shown that the quadrature sums that appear in applications of this method converge to the integral present in the integral equation. A method of solving the indicated equation numerically is given. It is shown that the specified class of solutions of integral equation for vortex layer intensity is distinguished only by the relative distribution of sets of discrete vortices and reference points.

The method of discrete vortices [1 — 3] provides a unique tool for investigating linear and nonlinear problems of steady and unsteady flow of inviscid incompressible fluid past thin lift airfoils. The continuous vortex layer which simulates the lift airfoil is replaced by a system of discrete vortices. Points, referred to as design points, at which conditions of impenetrability are satisfied are selected on the lift airfoil. The problem reduces to solving a system of linear algebraic equations for unknown circulations of discrete vortices.

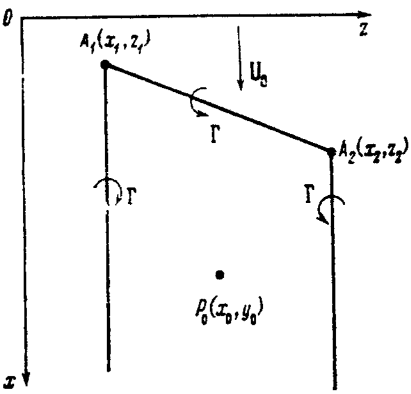

Fig. 1

The considered problem of flow has several solutions distinguished by the behavior of the vortex layer intensity in the vicinity of [the airfoil] edges. The sought solution is obtained by suitable selection of the relative position of sets of discrete vortices and design points. The mathematical basis of application of this method in the linear problem of steady flow past a thin wing of infinite span was given in [4] for circulation and in [5] for circulation-free flows.

Let us consider the oblique horseshoe vortex Π_{12} of constant intensity Γ consisting of the attached vortex (A_1, A_2) whose ends A_1 and A_2 are determined in the xyz-system (direction of the Ox-axis coincides with that of the unperturbed stream) by coordinates $(x_1, 0, z_1)$ and $x_2, 0, z_2)$ respectively, and of two rectilinear free vortices $(A_1, +\infty)$ and $(A_2, +\infty)$ running off the ends of the attached vortex and directed along the unperturbed stream velocity U_0 (Fig. 1). We write the equation of the vortex line (A_1, A_2) as $x(z) = a + zb$. Using the Biot-Savart formula we obtain for velocity V_{12} induced by vortex Π_{12} at point $P_0(x_0, z_0)$ the formula

$$V_{12} = \Gamma\,(v^2 - v^1)\,/\,(4\pi) \tag{1}$$
$$v^k = (\lambda + [(x_0 - x_k)^2 + (z_0 - z_k)^2]^{1/2})/[\lambda\,(z_0 - z_k)], \qquad k = 1, 2$$
$$\lambda = x_0 - a - z_0 b$$

Let us now consider the problem of flow past a canonical trapezoid which may be considered to be a half-wing whose edges are defined by $z = 0$ and $z = b$, which we shall call side edges, while the edges defined by equations $x_-(z) = a^\circ + zb^\circ$ and $x_+(z) = a^1 + zb^1$, will be called the leading and trailing edges, respectively. Velocity U_0 of the stable stream at some distance from the canonical trapezoid σ is directed along the positive axis Ox.

The rectangular wing is a particular case of the canonical trapezoid σ. One of its side edges may be reduced to a point.

We represent the lift vortex layer on σ by a system of oblique horseshoe vortices [1]. Let $D = [0, 1] \times [0, l]$ be a rectangle in the plane Ox^1z. We divide segment $[0, 1]$ of the Ox^1-axis in $n + 1$ segments $[x_i^1, x_{i+1}^1]$ $(i = 0, 1, \ldots, n)$ of length h_1, and segment $[0, l]$ of the Oz-axis in N segments $[z_k, z_{k+1}]$ $(k = 1, 2, \ldots, N)$ of length h_2. Segments $[x_j^1, x_{j+1}^1]$ and $[z_m, z_{m+1}]$ are divided in half by points x_{0j}^1 and z_{0m}, respectively.

Let us consider the mapping F of rectangle D on σ, defined by formulas

$$x\,(x^1, z) = x^1\,[x_+(z) - x_-(z)] + x_-(z), \quad z = z \tag{2}$$

The Jacobian of mapping F is of the form $J(z) = x_+(z) - x_-(z)$.

Let $A_{ik}(x_{ik}, z_k)$, $A_{im}(x_{im}, z_{0m})$ and $A_{jm}(x_{jm}, z_{0m})$ be the respective images of points

$$A_{ik}^1(x_i^1, z_k), \; A_{im}^1(x_i^1, z_{0m}), \; A_{jm}^1(x_{0j}^1, z_{0m})$$

Let us consider the oblique vortex Π_{ik} with attached vortex $(A_{ik}, A_{ik} + 1)$. Since $x\,(x^1, z) = a\,(x^1) + zb\,(x^1)$, $a\,(x^1) = a^\circ + x^1\,(a^1 - a^\circ)$, and $b\,(x^1) = b^\circ + x^1\,(b^1 - b^\circ)$, the equation of the line of the attached vortex $(A_{ik}, A_{i,k+1})$ is of the form $x_i(z) = x\,(x_i^1, z)$. The intensity of vortex Π_{ik} is determined by formula $\Gamma_{ik} = \varphi_{ik}h_1$, $\varphi_{ik} = J(z_{0k})\,\gamma\,(x_{im}, z_{0m})$, $i = 1, \ldots, n$, $k = 1, \ldots, N$. In this formula $\gamma\,(x, z)$ is the component on axis z of the vortex layer intensity at point $A\,(x, z)$ of the canonical trapezoid σ. The velocities induced by oblique vortices are determined at the design points $P_{jm}(x_{jm}, z_{0m})$ that are images of points $P_{jm}^1(x_{0j}^1, z_{0m})$. We denote the velocity induced by the oblique vortex Π_{ik} at point P_{jm} by V_{jm}^{ik}. Velocity V_{jm} induced at point P_{jm} by the complete system of oblique vortices is determined in conformity with formula (1) and relationship $x_{jm} = a\,(x_{0j}^1) - z_{0m}b\,(x_{0j}^1)$ by formula

$$4\pi V_{jm} = \sum_{l=1}^{3} C_l, \quad C_1 = \sum_{i=1}^{n} \varphi_{i1}\Lambda_{jm}^{i1}\,h_1 \tag{3}$$
$$C_2 = \sum_{i=1}^{n}\sum_{k=2}^{N} (\varphi_{ik} - \varphi_{i,k-1})\,\Lambda_{jm}^{ik}\,h_1, \quad C_3 = \sum_{i=1}^{n} \varphi_{iN}\Lambda_{jm}^{i,N+1}h_1$$
$$\Lambda_{jm}^{ik} = \{\lambda_{jmi} + [(x_{jm} - x_{ik})^2 + (z_{0m} - z_k)^2]^{1/2}\}/[\lambda_{jmi}\,(z_{0m} - z_k)]$$
$$\lambda_{jmi} = x_{jm} - a\,(x_i^1) - z_{0m}b\,(x_{0j}^1) = J\,(z_{0m})\,(x_{0j}^1 - x_i^1)$$

Let us assume that function $\partial\gamma\,(x\,(x^1, z), z)\,/\,\partial z$ belongs to class H^* on the rectangle $D = [0, 1] \times [0, l]$ of plane Ox^1z. Function $\psi^*(x^1, z)$ belongs to class H^* [5]

on the rectangle $[a, b] \times [c, d]$ if it is of the form $\psi^* (x^1, z) = \psi (x^1, z) \times (x^1 - a)^{-\nu_1} (b - x^1)^{-\mu_1} (z - c)^{-\nu_2} (d - z)^{-\mu_2}$, $0 \leqslant \nu_k, \ \mu_k < 1, \ k = 1,2$ and $\psi (x^1, z) \in H$ on $[a, b] \times [c, d]$, i.e. it satisfies the Hölder condition over the totality of variables [6]. Then applying the results of [4] and extending those for the two-dimensional singular integral of the Cauchy type [5], we find that for $n \to \infty$ and $N \to \infty$ with $0 < \delta \leqslant h_1 / h_2 \leqslant T < +\infty$, equality (3) for velocity $V (x_0, z_0)$ at point $P_0 (x_0, z_0)$ assumes the form

$$4\pi V (x_0, z_0) = \sum_{l=1}^{3} I_l \tag{4}$$

$$I_1 = \int_0^1 \Lambda (\lambda, x, x_0, 0, z_0)\, \varphi (x, 0)\, dx^1$$

$$I_2 = \int_0^1 \int_0^l \Lambda (\lambda, x, x_0, z, z_0) \left[\frac{\partial \varphi (x, z)}{\partial z} \right] dx^1 dz$$

$$I_3 = \int_0^1 \Lambda (\lambda, x, x_0, l, z_0)\, \varphi (x, l)\, dx^1$$

$$\Lambda (\lambda, x, x_0, z, z_0) = \{\lambda + [(x_0 - x)^2 + (z_0 - z)^2]\}^{1/2} / [\lambda (z_0 - z)]$$
$$\lambda = J (z_0)(x_0^1 - x^1), \quad x_0 = x (x_0^1, z_0), \quad x = x (x^1, z)$$

Applying now the concept of integral in Adamard's meaning of the finite part [7], taking into account the relation $\varphi (x, z) = \gamma (x (x^1, z), z) J (z)$, and passing to variables x and z using the substitution of variables defined by mapping F, for the intensity $\gamma (x, z)$ $((x_0, z_0) \in \sigma)$ we obtain the equation

$$\frac{1}{4\pi} \iint_{(\sigma)} \gamma (x, z) \frac{1}{(z_0 - z)^2} \left(1 + \frac{x_0 - x}{\sqrt{(x_0 - x)^2 + (z_0 - z)^2}} \right) dx\, dz = V (x_0, z_0) \tag{5}$$

To determine the numerical value of intensity $\gamma (x, z)$ of the vortex layer at design points we consider the system of linear algebraic equations

$$\sum_{i=1}^{n} \sum_{k=1}^{N} V_{jm}^{ik} = V_{jm}, \quad j = 1, \ldots, n, \quad m = 1, \ldots, N \tag{6}$$

where V_{jm} is assumed known at points P_{jm} on the basis of the condition of impenetrability at surface σ. The sets of vortices $\{(A_{ik}, A_{i,k+1}), i = 1, \ldots, n;\ k = 1, \ldots, N\}$ and design points $\{P_{jm}, j = 1, \ldots, n, m = 1, \ldots, N\}$ are arranged in such manner that the vortices are closest to the leading edge and the design points closest to the trailing edge.

For analyzing the behavior of $\gamma (x, z)$ close to the leading and trailing edges we rewrite system (6) with allowance for (3) in the form

$$-2 \sum_{i=1}^{n} \gamma (x_{im}, z_{0m})\, \theta (x_i^1) \frac{h_1}{x_{0j}^1 - x_i^1} = 4\pi V_{jm} + S_{jm} \tag{7}$$

$$S_{jm} = \sum_{l=1}^{3} S^l, \quad S^1 = \sum_{i=1}^{n} \varphi_{i1} (\Lambda_{jm}^{i1} - \Phi_{jm}^{i1})\, h_1$$

$$S^2 = \sum_{i=1}^{n} \sum_{k=2}^{N} (\varphi_{ik} - \varphi_{i,k-1}) (\Lambda_{jm}^{ik} - \Phi_{jm}^{ik})\, h_1$$

$$S^3 = \sum_{i=1}^{n} \varphi_{iN} (\Lambda_{jm}^{i,N+1} - \Phi_{jm}^{i,N+1}) h_1$$

$$\Phi_{jm}^{ik} = | z_{0m} - z_k | \theta (x_i^1) / [\lambda_{jmi} (z_{0m} - z_k)]$$

$$\theta^2 (x_i^1) = 1 + b^2 (x_i^1), \quad j = 1, \ldots, n; \quad m = 1, \ldots, N$$

According to [4] for each fixed m we have

$$2\gamma (x_{im}, z_{0m}) \theta (x_i^1) = \frac{1}{\pi^2} \sqrt{\frac{1 - x_i^1}{x_i^1}} \sum_{j=1}^{N} \sqrt{\frac{x_{0j}^1}{1 - x_{0j}^1}} \times \tag{8}$$

$$(4\pi V_{jm} + S_{jm}) \frac{h_1}{x_i^1 - x_{0j}^1} + \alpha (i, m); \quad i = 1, \ldots, n, \quad m = 1, \ldots, N$$

where $\alpha (i, m) \to 0$ uniformly for all $(x_i^1, z_{0m}) \in [\delta, 1 - \delta] \times [\delta, l - \delta]$ and any number $\delta > 0$.

System (8) shows that when system (6) has a solution, function $\gamma (x, z)$ which is the limit of that solution for $n \to \infty$ and $N \to \infty$ satisfies for any $z \in (0, l)$ the relations

$$\gamma (x (x^1, z), z) |_{x^1=1} = \gamma (x_+ (z), z) = 0$$

$$\gamma (x (x^1, z), z) |_{x^1=0} = \gamma (x_- (z), z) = 0$$

Let us now consider as the lift airfoil σ a wing of complex plan form with straight edges or a schematized aircraft [3].

We assume that σ lies in the Oxz -plane, and draw through the contour corner points straight lines parallel to the Ox -axis. The total surface σ is then divided into canonical trapezoids σ_ε, $\varepsilon = 1, \ldots, p$ which intersect only along the side edges. Let the side edges of σ_ε be defined by the equations $z = l_\varepsilon^1$ and $z = l_\varepsilon^2$, and the leading and trailing edges by the equations $x_-^\varepsilon (z) = a_\varepsilon^\circ + z b_\varepsilon^\circ$, and $x_+^\varepsilon (z) = a_\varepsilon^1 + z b_\varepsilon^1$, $\varepsilon = 1, \ldots, p$, respectively.

Let us consider p specimens of planes $Ox^\varepsilon z$ and of rectangles $D_\varepsilon = [0, 1] \times [l_\varepsilon^1, l_\varepsilon^2]$ in each of these. We divide segment $[0, 1]$ of the Ox^ε -axis by points x_i^ε, $x_{0i}^\varepsilon (i = 1, \ldots, n_\varepsilon$ for a given $\varepsilon)$ at pitch h_1^ε and segment $[l_\varepsilon^1, l_\varepsilon^2]$ by points z_k^ε, z_{0k}^ε $(k = 1, \ldots, N_\varepsilon)$ at pitch h_2^ε. If σ_ε and σ_ν lie along the stream behind each other, i.e. $l_\varepsilon^k = l_\nu^k$, $k = 1, 2,$ we set $h_2^\varepsilon = h_2^\nu$ and, consequently, $N_\varepsilon = N_\nu$. The last condition is imposed in order to have coincidence of lines of free vortices running off σ_ε and σ_ν.

The oblique vortices Π_{ik}^ε on trapezoid σ_ε are determined using mapping F_ε of the rectangle D_ε on σ_ε, defined by relations $x (x^\varepsilon, z) = x^\varepsilon [x_+^\varepsilon (z) - x_-^\varepsilon (z)] + x_-^\varepsilon (z)$ and $z = z$. The Jacobian of transformation F_ε is of the form $J_\varepsilon (z) = x_+^\varepsilon (z) - x_-^\varepsilon (z)$.

Taking into account the formula

$$V_{12} = \Gamma w (x (z), x_0, z, z_0) / (4\pi) \tag{9}$$

$$w (x (z), x_0, z_1, z_2, z_0) = \int_{z_1}^{z_2} R (x (z), x_0, z, z_0)\, dz$$

$$R (x (z), x_0, z, z_0) = \frac{1}{(z_0 - z)^2} \left(1 + \frac{x_0 - x (z)}{\sqrt{[x_0 - x (z)]^2 + (z_0 - z)^2}} \right)$$

which follows from (1) we obtain for velocity V_{jm}^ε induced at point P_{jm}^ε $(x_{jm}^\varepsilon, z_{0m}^\varepsilon)$

of trapezoid σ_ε by the total system of oblique vortices on σ the formula

$$V_{jm}^{\varepsilon} = V_{jm}^{\varepsilon\varepsilon} + \sum_{\nu=1,\,\nu\neq\varepsilon}^{p} \sum_{i=1}^{n_\nu} \sum_{k=1}^{N_\nu} \frac{\Gamma_{ik}^{\nu}}{4\pi} w_{ikjm}^{\nu\varepsilon} \tag{10}$$

$$w_{ikjm}^{\nu\varepsilon} = w\,(x_i^{\nu}(z),\ x_{jm}^{\varepsilon},\ z_k^{\nu},\ z_{k+1}^{\nu},\ z_{0m}^{\varepsilon}),\quad x_i^{\nu}(z) = x\,(x_i^{\nu},\ z)$$

where $V_{jm}^{\nu\varepsilon}$ is the velocity induced at point P_{jm}^{ε} of trapezoid σ_ε by the system of discrete vortices on trapezoid σ_ν, and $x_i^{\nu}(z)$ is the equation of the line of vortex $(A_{ik}^{\nu},\ A_{i,k+1}^{\nu})$ attached to that trapezoid whose end coordinates are $(x_{ik}^{\nu} = x_i^{\nu}(z_k),\ z_k)$ and $(x_{ik+1}^{\nu} = x_i^{\nu}(z_{k+1}),\ z_{k+1})$.

Let us assume that function $\gamma\,(x\,(x^{\varepsilon}, z), z)$ belongs to class H^* on rectangle D_ε. Applying to $V_{jm}^{\varepsilon\varepsilon}$ a reasoning similar to that used previously in the case of the canonical trapezoid, noting that for $\nu \neq \varepsilon$ either $z_0 - z$ or $\lambda_\varepsilon^{\nu} = x\,(x_0^{\varepsilon},\ z_0) - a\,(x^{\nu}) - z_0 b\,(x^{\nu})$ do not change signs for $z_0 \in (l_\varepsilon^1, l_\varepsilon^2)$ and $z \in [l_\nu^1, l_\nu^2]$, and passing to limit for $n_\varepsilon \to \infty$, $N_\varepsilon \to \infty$, $0 < \delta \leqslant h_1^{\varepsilon} / h_2^{\varepsilon}$, $h_k^2 / h_k^{\nu} \leqslant T < +\infty, \varepsilon, \nu = 1, \ldots, p$; $k = 1, 2$, for the velocity $V\,(x_0, z_0)$ at point $P_0\,(x_0, z_0)$ of the lift airfoil we obtain

$$V\,(x_0, z_0) = \frac{1}{4\pi} \sum_{\nu=1}^{p} \iint_{(\sigma_\nu)} \gamma\,(x, z)\, R\,(x, x_0, z, z_0)\, dx\, dz \tag{11}$$

Formula (11) implies that Eq. (5) is also valid for the considered lift airfoil σ of complex shape.

To determine the numerical value of solution $\gamma\,(x, z)$ at design points P_{jm}^{ε}, $\varepsilon = 1, \ldots, p$ ($j = 1, \ldots, n_\varepsilon$, $m = 1, \ldots, N_\varepsilon$ for a given ε) of surface σ it is necessary to consider the system of linear algebraic equations

$$\sum_{\nu=1}^{p} \sum_{i=1}^{n_\nu} \sum_{k=1}^{N_\nu} V_{ikjm}^{\nu\varepsilon} = V_{jm}^{\varepsilon} \tag{12}$$

where $V_{ikjm}^{\nu\varepsilon}$ is the velocity induced by the oblique vortex Π_{ik}^{ν} of trapezoid σ_ν at point P_{jm}^{ε} of trapezoid σ_ε.

The properties of system (12) imply, as before, that function $\gamma\,(x, z)$ satisfies the relations

$$\gamma\,(x_+^{\varepsilon}(z), z) = 0,\quad \gamma\,(x_-^{\varepsilon}(z), z) = \infty,\quad z \in (l_\varepsilon^1,\ l_\varepsilon^2)$$

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